

THE LECTURE HALL THEOREM VIA ABACUS DIAGRAMS

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ABSTRACT. Bousquet-Mélou & Eriksson's lecture hall theorem generalizes Euler's celebrated distinct-odd partition theorem. We present an elementary and short proof of the lecture hall theorem using a simple bijection involving abacus diagrams.

1. INTRODUCTION

Lecture hall partitions were introduced by Bousquet-Mélou and Eriksson [BME97a] as sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfying

$$(1.1) \quad 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_i}{i} \leq \dots \leq \frac{\lambda_n}{n}.$$

Pictorially, the diagram of λ represents the heights of seats in a lecture hall with n rows. The requirement that each row be able to see the speaker (who is located at height zero) then corresponds to the slope condition given in the definition. In [BME97a], the following remarkable theorem was shown.

Theorem 1.1. (The Lecture Hall Theorem) *We have*

$$(1.2) \quad \sum_{\lambda} x^{|\lambda|} = \prod_{i=1}^n \frac{1}{1 - x^{2i-1}}$$

where the sum is taken over all lecture hall partitions λ with n parts and $|\lambda| = \sum_{i=1}^n \lambda_i$.

This can be viewed as a finite generalization of Euler's classical result that the number of partitions of a given integer having distinct parts is equal to the number of partitions of that integer having odd parts. To see this, observe that the lecture hall inequalities (1.1) imply that λ always has distinct parts. Conversely, if we are given any partition λ with distinct parts, then there exists an N for which the partitions of length $n > N$ obtained from λ by including parts of size zero all satisfy the lecture hall inequalities. In this sense, the left side of (1.2) becomes the generating function for partitions with distinct parts as $n \rightarrow \infty$, while the right side of (1.2) becomes the generating function for partitions with odd parts. Hence, we recover Euler's result. A gentle introduction to other generalizations of Euler's result can be found in [AE04, Chapter 9].

Bousquet-Mélou and Eriksson gave two proofs of Theorem 1.1 in [BME97a]: one relied on Bott's formula for the affine Weyl group \tilde{C} and the other was a relatively complicated recursive argument. Shortly thereafter they further refined Theorem 1.1 and gave the first truly bijective proof [BME99, §3] of the Lecture Hall Theorem. Other bijective proofs followed by Yee [Yee01, Yee02] which also proved the refined version, and by Eriksen [Eri02] whose construction gave further support to some open conjectures on generalized lecture hall partitions [BME97b]. Savage and Yee [SY08] also gave a new proof by studying the more general ℓ -sequences. These bijective proofs are elementary yet also somewhat involved.

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Our proof adds to this body of work by providing a bijection that is both elementary and straightforward. The proof boils down to three pictures (see Examples 2.1, 3.3, and 4.3) involving abacus diagrams, each of which are simple and intuitive. It remains to be seen if our streamlined proof lends itself to the refined or generalized versions of Theorem 1.1. It would also be interesting to determine whether our bijection agrees with one from the literature above.

Many proofs of Theorem 1.1 are short yet rely heavily on background knowledge of some external theory. For example, the other proof of [BME99] relied on a q -analog of Bott's formula for \tilde{C} and MacMahon's partition analysis was utilized by Andrews in [And98]. Other proofs by Savage et al. use q -series [CS04, ACS09]. There has also been extensive work done by Savage and others [CLS07] on understanding the geometry of lecture hall partitions as lattice points in the cone given by the inequalities that define those partitions. The recursive proof in [BME97a] can be interpreted in these geometric terms but an honest proof of the lecture hall theorem in this lattice point sense currently remains out of reach.

In this article, we will develop abacus diagrams from scratch as a natural way to encode lecture hall partitions. Abacus diagrams were originally introduced by James [JK81] to study the modular representation theory of the finite symmetric group. These "type A " diagrams correspond to core partitions and have been used by Wildon [Wil08] and Garvin–Kim–Stanton [GKS90] to study the partition function. The abacus diagrams in our work have appeared previously [HJ12] as minimal length coset representatives in the affine Weyl group \tilde{C} , and correspond to symmetric core partitions. We do not rely on these connections in our work.

The rest of the paper constitutes our proof of Theorem 1.1. In Section 2 we explain how to encode lecture hall partitions as abacus diagrams. In Section 3, we show that the abacus diagrams are also in bijection with certain partitions whose parts are bounded. (This fact was shown previously in [HJ12], but we include a proof here to be self-contained.) It is straightforward to verify that the generating function for these bounded partitions is the same one that appears in the Lecture Hall Theorem. Finally, we show in Section 4 that the composite bijection from lecture hall partitions to bounded partitions preserves the sum-of-parts statistic. This shows that the lecture hall partitions have the same generating function as the bounded partitions, and completes the proof of the Lecture Hall Theorem.

2. ABACUS DIAGRAMS FOR LECTURE HALL PARTITIONS

Fix a positive integer n . In our work, we use a particular type of diagram to encode the lecture hall partitions of length n , which we now describe. We begin with an array having $2n$ columns and countably many rows. We label the entry in the i th row and j th column of the array by the integer $j + 2ni$, where $1 \leq j \leq 2n$. In figures, we will draw the rows increasingly down the page, and columns increasingly from left to right. Then these labels linearly order the entries of the array, which we refer to as **reading order**. We also say that column j is **dual** to column $2n + 1 - j$, and we call the entries $\{1 + nk, 2 + nk, \dots, (n - 1) + nk, n(k + 1)\}$ the k th **window** of the array. To create our diagram, we highlight certain entries in the array; such entries are called **beads** and will be circled in figures. Entries that are not beads will be called **gaps**.

To encode a lecture hall partition λ , we begin with the largest part λ_n , and set entry λ_n in the array to be a bead b_n . Next, skipping entries that lie in the column containing b_n or its dual column, we count out λ_{n-1} positive positions and place a bead b_{n-1} . Continuing in this way, we place one bead b_i for each part λ_i by counting out λ_i positive entries, not including the entries of any column containing a previously placed bead b_j for $j > i$, nor the duals of such columns. If $\lambda_i = 0$, then a bead b_i is placed at the largest nonpositive entry in a column that does not contain a previously placed bead, nor the dual of a column containing a previously placed bead. We will refer to these beads b_i as **defining beads**.

In order to complete the diagram, we perform two additional steps for each defining bead b_i . First, we create beads at all of the entries above b_i lying in the same column as b_i . Second, if b_i has label j ,

then the entry labeled $1 - 2n - j$ occurs in the dual column to b_i . We create beads at this entry, and all entries lying above it in the dual column to b_i . All of the other entries in the diagram are gaps. We call this completed diagram the **abacus diagram for λ** .

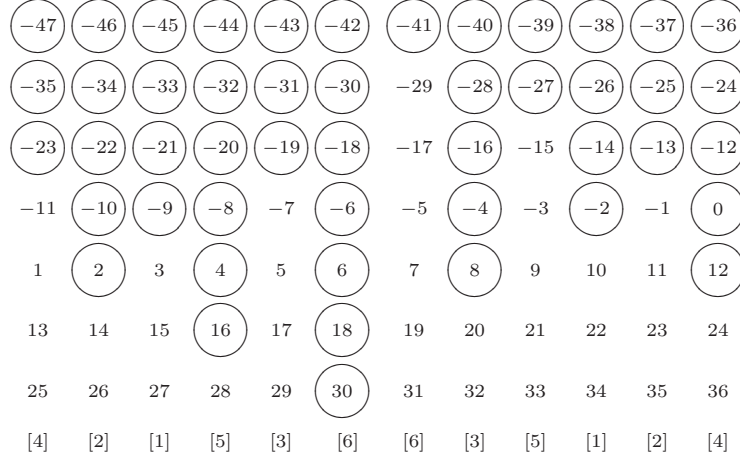


FIGURE 1. Abacus diagram for $\lambda = (0, 1, 4, 8, 14, 30)$.

Example 2.1. Let $n = 6$. Then $\lambda = (0, 1, 4, 8, 14, 30)$ is a lecture hall partition since

$$0 \leq \frac{0}{1} \leq \frac{1}{2} \leq \frac{4}{3} \leq \frac{8}{4} \leq \frac{14}{5} \leq \frac{30}{6}.$$

Part of the abacus diagram for λ is shown in Figure 1; the unseen negative entries are all beads and the unseen positive entries are all gaps. The defining beads are $b_6 = 30$, $b_5 = 16$, $b_4 = 12$, $b_3 = 8$, $b_2 = 2$, and $b_1 = -2$. These beads lie in windows 4, 2, 1, 1, 0, and -1 , respectively.

For each $1 \leq i \leq n$, we say that entries in the column containing b_i and in the dual column have **class i** . A position p in the abacus diagram for λ is **i -active** if p lies weakly between position 1 and the position of the defining bead b_i in reading order, and if the class of p is less than or equal to the class of b_i . Then we can summarize our construction as:

- (2.1) The abacus diagram for λ is constructed by placing defining beads so that there are λ_i positions that are i -active, for each $1 \leq i \leq n$.

Example 2.2. In Figure 1, the classes of each column are indicated in brackets. The 4-active positions are 12, 11, 10, 8, 5, 3, 2, 1; there are $\lambda_4 = 8$ of these.

To describe the inverse construction, we will also consider arbitrary collections of beads in the array. We will say that such a collection of beads forms an **abacus diagram** if

- No bead in any column is preceded in reading order by a gap in that column; when this condition holds, we say that the beads in the diagram are **flush**.
- A bead occurs in position j if and only if a gap occurs in position $1 - j$ for all $j \in \mathbb{Z}$; when this condition holds, we say that the beads in the diagram are **balanced**.

From the set consisting of the lowest bead in each column, the last n of these beads in reading order will be called the **defining beads** of the abacus diagram. Observe that no two defining beads lie in dual columns, and that any such set of defining beads determines a unique balanced flush abacus.

It is straightforward to verify that the abacus diagrams produced from lecture hall partitions are abacus diagrams as defined in the preceding paragraph. Moreover, we can recover a lecture hall partition from an arbitrary abacus diagram by counting the number of i -active positions prior to each defining bead in the diagram.

We claim that this is a bijection.

Theorem 2.3. *The lecture hall partitions are in bijection with abacus diagrams via the constructions given above.*

Proof. Composing the constructions, in either order, recovers the original object. Hence, it suffices to prove that the inequalities defining the lecture hall partitions are equivalent to the conditions defining the abacus diagrams.

For all i , we have

$$\frac{\lambda_i}{i} \leq \frac{\lambda_{i+1}}{i+1} \quad \text{if and only if} \quad \lambda_i \leq \lambda_{i+1} - \frac{\lambda_{i+1}}{i+1},$$

which is equivalent to

$$(2.2) \quad \lambda_i \leq \lambda_{i+1} - \lceil \frac{\lambda_{i+1}}{i+1} \rceil,$$

since the parts of λ must be integers.

Under the correspondence (2.1), each positive window prior to the window containing the $(i+1)$ st defining bead will have exactly $i+1$ positions that are $(i+1)$ -active. Therefore, $\lceil \frac{\lambda_{i+1}}{i+1} \rceil$ represents the window containing the $(i+1)$ st defining bead. Hence, the inequality in (2.2) means that the maximum number of i -active positions is the number of $(i+1)$ -active positions minus one position from each positive window up to and including the window containing the $(i+1)$ st defining bead. This difference is equivalent to the construction we have given, in which the entries of class $i+1$ and all higher classes are ignored when placing b_i so that there are λ_i positive i -active positions. In particular, the defining beads b_1, b_2, \dots, b_n occur in the abacus diagram in reading order. \square

3. BOUNDED PARTITIONS FROM ABACUS DIAGRAMS

We say that a partition \mathbf{p} is **bounded** if all of its parts are at most $2n$ and those parts less than or equal to n are distinct. In contrast to the lecture hall partitions, these partitions are straightforward to enumerate: if we let $|\mathbf{p}|$ denote the sum of the parts of \mathbf{p} then we obtain the generating function

$$\sum_{\text{bounded partitions } \mathbf{p}} x^{|\mathbf{p}|} = \frac{(1+x)(1+x^2) \cdots (1+x^n)}{(1-x^{n+1})(1-x^{n+2}) \cdots (1-x^{2n})} = \prod_{i=1}^n \frac{1}{1-x^{2i-1}}.$$

We claim that each abacus diagram corresponds to a unique bounded partition. Consider an abacus diagram whose positive beads occur in positions $\{\dot{b}_1, \dot{b}_2, \dots, \dot{b}_k\}$; note that these are determined by the defining beads, but we include all positive beads in this list. We form the partition \mathbf{p} whose distinct parts consist of the positions of those beads lying in the 0th window of the array, and for every positive bead \dot{b}_i lying outside the 0th window we include a part \mathbf{p}_i of size

$$\#(\text{gaps between } \dot{b}_i - 2n \text{ and } \dot{b}_i) + 1.$$

Example 3.1. In Figure 1, we have beads in positions 2, 4, and 6 lying in the 0th window of the abacus diagram, so these are the distinct parts less than or equal to $n = 6$. The bead in position 8 has 6 gaps lying between itself and the bead in position -4 . Similarly, there are 7 gaps lying between beads 12 and

0; 8 gaps lying between beads 16 and 4; 8 gaps lying between beads 18 and 6; and 11 gaps lying between beads 30 and 18. Therefore, the corresponding bounded partition is

$$\mathbf{p} = (2, 4, 6, 7, 8, 9, 9, 12).$$

Theorem 3.2. *The abacus diagrams are in bijection with the bounded partitions via the construction above.*

Proof. We first show that the construction is well-defined. Clearly if \dot{b}_i is in the 0th window then its corresponding part is between 1 and n . Otherwise, \dot{b}_i is a bead in a position greater than n , and we have that $\dot{b}_i - 2n$ is also a bead because abacus diagrams are flush. There is one other position between $\dot{b}_i - 2n$ and \dot{b}_i that is in the same class as \dot{b}_i , and this position must be a gap since abacus diagrams are balanced. The other $2n - 2$ positions all belong to the other $n - 1$ classes. Each class has precisely two positions, at least one of which is a gap since abacus diagrams are balanced. Hence, the number of gaps between $\dot{b}_i - 2n$ and \dot{b}_i must be at least n . On the other hand, since there are $2n - 1$ positions lying strictly between $\dot{b}_i - 2n$ and \dot{b}_i , there can be at most $2n - 1$ gaps between them. Hence, each part p_i that we append satisfies $n + 1 \leq p_i \leq 2n$, as required. Thus, \mathbf{p} is a composition of parts between 1 and $2n$ with those parts between 1 and n being distinct. The flush condition also implies that the number of gaps between $\dot{b}_i - 2n$ and \dot{b}_i is increasing as a function of \dot{b}_i 's position. Hence, if we append the parts p_i following the reading order of the beads \dot{b}_i , then \mathbf{p} will be sorted increasingly so \mathbf{p} is a bounded partition.

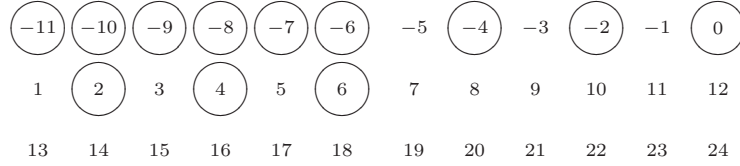
Next, we give the inverse construction. To encode a bounded partition $\mathbf{p} = (p_1, \dots, p_s, p_{s+1}, \dots, p_t)$, where $1 \leq p_1 < p_2 < \dots < p_s \leq n$ are the distinct parts, begin by placing beads in positions p_1, p_2, \dots, p_s and leave all other positions in window 0 as gaps. Next, place beads and gaps in window -1 by leaving the positions $1 - p_1, 1 - p_2, \dots, 1 - p_s$ as gaps and assigning beads to all other positions in window -1 .

To facilitate the rest of the construction, we say that a position j in an abacus diagram is **supported** if the position $j - 2n$ is a bead, and we say that j is **unsupported** otherwise. Having placed the first $i - 1$ beads so that each bead accurately encodes a part of the bounded partition, we claim that there is a unique position for a new bead \dot{b}_i such that there exist exactly $p_i - 1$ gaps between \dot{b}_i and $\dot{b}_i - 2n$, and so that the resulting abacus diagram remains flush.

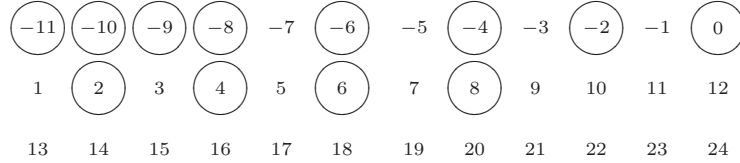
To see this, imagine placing a new positive bead \dot{b}_i in the position just after the last bead \dot{b}_{i-1} in reading order, or at position $n + 1$ if $i = s + 1$. Then the number of gaps between \dot{b}_i and $\dot{b}_i - 2n$ is exactly $p_{i-1} - 1$, or simply n if $i = s + 1$. Next, consider moving \dot{b}_i forward in reading order one entry at a time. Each time we pass an unsupported position j , we lose one gap from position $j - 2n$ but we gain a gap at position j , so the number of gaps between \dot{b}_i and $\dot{b}_i - 2n$ is unchanged. As we pass a supported position j , we only gain the gap at position j so the number of gaps between \dot{b}_i and $\dot{b}_i - 2n$ increases by 1.

In order to both create the correct number of gaps and to have a flush abacus, we must therefore place \dot{b}_i at the $(p_i - p_{i-1} + 1)$ st next supported position after \dot{b}_{i-1} , or at the $(p_i - n)$ th next supported position after position n in the case that $i = s + 1$. Since the number of supported positions between \dot{b}_{i-1} and $\dot{b}_{i-1} + 2n$ remains equal to $2n - p_{i-1} + 1$, this is always possible. This construction determines the bead/gap status of every position greater than or equal to $-n$, and we complete the construction by forming the unique balanced abacus that agrees with these entries. \square

Example 3.3. Given the bounded partition $\mathbf{p} = (2, 4, 6, 7, 8, 9, 9, 12)$, we form the partial abacus diagram consisting of the distinct parts $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = 4$, and $\mathbf{p}_3 = 6$:



The supported positions are 8, 10, 12, 14, 16 and 18. These positions would correspond to bounded parts of size 7, 8, 9, 10, 11 and 12, respectively, as we can see by counting the number of gaps between each entry and the corresponding entry in the previous row. Since $\mathbf{p}_4 = 7$, we must place the next bead in position 8, obtaining:



Now the supported positions are 10, 12, 14, 16, 18 and 20. These correspond to bounded parts of size 7, 8, 9, 10, 11 and 12, respectively. Since $\mathbf{p}_5 = 8$, we must place the next bead in position 12. Continuing in this fashion, and then setting the bead/gap status of the negative entries to balance with the positive entries we have specified, we obtain the abacus in Figure 1.

4. THE BIJECTIONS PRESERVE THE SUMS OF PARTS

Fix an abacus diagram on $2n$ columns. We know from the previous sections that there is a unique lecture hall partition λ with n parts and a corresponding bounded partition \mathbf{p} . To show that the lecture hall partitions are enumerated by the same generating function as the bounded partitions, it suffices to show that $|\lambda| = |\mathbf{p}|$. In the running example from Figure 1, we have

$$|\lambda| = 0 + 1 + 4 + 8 + 14 + 30 = 57 = 2 + 4 + 6 + 7 + 8 + 9 + 9 + 12 = |\mathbf{p}|.$$

The proof presented here is a straightforward induction proof, inducting on the number of positive beads lying beyond position n in an abacus. It does need a preliminary technical lemma. As before, the terms b_i are (positions of) the defining beads for class i in a given abacus and, by Theorem 2.3, if $i < k$ then $b_i < b_k$. Let $c(a)$ denote the class of position a , and $\{a < b : c(a) = k\}$ be the set of positive positions less than b of class k .

Lemma 4.1. *Fix an abacus diagram with defining beads b_1, b_2, \dots, b_n . If $i < k$ with $b_k - b_i < 2n$ then $\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\} = 1$.*

Proof. Denote the window that contains b_i as the ω_i -th window. Since b_k comes after b_i in reading order then in every window previous to the ω_i -th window there is exactly one position of class i and one of class k . On the other hand, since $b_k - b_i < 2n$ then b_k is in one of the ω_i -th window, $(\omega_i + 1)$ -th window or the $(\omega_i + 2)$ -th window. Consequently, the difference $\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\}$ can be restricted to those positions in the ω_i -th, $(\omega_i + 1)$ -th and $(\omega_i + 2)$ -th windows. In the expression below we only count positions in these three windows.

The difference $\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\}$ can be expanded as

$$\begin{aligned} & \underbrace{(\#\{a < b_i : c(a) = i\} - \#\{a < b_i : c(a) = k\})}_{=0} \\ & + \underbrace{\#\{a = b_i : c(a) = i\}}_{=1} + \#\{a : b_i < a < b_k, c(a) = i\}. \end{aligned}$$

Case (i): the position of class k in ω_i occurs before b_i . In this case b_k must be in the class k position of either the $(\omega_i + 1)$ -th window or the $(\omega_i + 2)$ -th window, the latter window having the same class positions as ω_i , in the former reversed. Either way the class i position in the $(\omega_i + 1)$ -th window is the only position in the set $\{a : b_i < a < b_k, c(a) = i\}$. Also, $\{a < b_i : c(a) = k\}$ has only one element, the class k position in window ω_i . Hence, we have $\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\} = (0 - 1) + 1 + 1 = 1$.

Case (ii): the position of class k in ω_i occurs after b_i . In a similar fashion to Case (i), we clearly have $\#\{a < b_i : c(a) = k\} = 0$. Since b_k is either in a position after b_i in the ω_i -th window or in a position before the class i position in the $(\omega_i + 1)$ -th window, the set $\{a : b_i < a < b_k, c(a) = i\}$ is empty and $\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\} = (0 - 0) + 1 + 0 = 1$. \square

Theorem 4.2. *For every abacus diagram, the corresponding lecture hall partition λ and the corresponding bounded partition \mathbf{p} satisfy $|\lambda| = |\mathbf{p}|$.*

Proof. We will prove the statement by induction on the number of positive beads t lying beyond window 0 in an abacus. If $t = 0$ then the bounded partition \mathbf{p} corresponding to the abacus contains only distinct parts. Since all the parts of the lecture hall partition λ correspond to positions in the abacus between 1 and n , we have that $\lambda = \mathbf{p}$.

Next, suppose we have an initial abacus with $t - 1$ positive beads lying beyond position n , and let us assume our inductive hypothesis that $|\lambda| = |\mathbf{p}|$ for this initial abacus. Let us call this the $(t - 1)$ -abacus, the prefix representing the number of positive beads lying beyond position n in the abacus. Placing an additional positive bead in the $(t - 1)$ -abacus to create a t -abacus diagram that is balanced and flush means that we can only place a bead directly below an already existing defining bead. Assume this bead is b_i , the defining bead of class i in the $(t - 1)$ -abacus, and so the new bead is in position $b_i + 2n$. Without loss of generality, we can assume that $b_i + 2n$ is the last bead in reading order in the t -abacus – if it were not we could remove the last bead in reading order to attain another abacus with $t - 1$ beads and assume the induction hypothesis on this abacus. All other positions remain as beads or gaps as in the initial abacus but note that the classes of the columns have changed. In particular,

- (a) the bead b_i that was the defining bead of class i in the $(t - 1)$ abacus it is now of class n . It is no longer a defining bead, rather $b_i + 2n$ is the defining bead of class n in the t -abacus. As a consequence, positions of class i in the $(t - 1)$ -abacus are of class n in the t -abacus;
- (b) the defining beads $b_{i+1}, b_{i+2}, \dots, b_{n-1}, b_n$ in the $(t - 1)$ -abacus all lie between b_i and $b_i + 2n$ in reading order in the t -abacus. They remain defining beads in the t -abacus but their classes are now shifted down by 1. That is, b_k is of class $k - 1$ in the t -abacus for all $i + 1 \leq k \leq n$. As a consequence, positions of class k in the $(t - 1)$ -abacus are of class $k - 1$ in the t -abacus;
- (c) the defining beads b_1, \dots, b_{i-1} of the $(t - 1)$ -abacus are all in positions less than b_i and are defining beads for the same respective classes in the t -abacus.

Let λ^* and \mathbf{p}^* denote the lecture hall and bounded partitions respectively of the t -abacus. Note that $\mathbf{p}^* = \mathbf{p} + \mathbf{p}_t^*$ where the part \mathbf{p}_t^* is created by the new bead $b_i + 2n$. Since the beads b_{i+1}, \dots, b_n are all strictly between b_i and $b_i + 2n$ and since all other defining beads are less than b_i then the number of gaps between b_i and $b_i + 2n$ is $(2n - 1) - (n - i) = n + i - 1$. Hence the part $\mathbf{p}_t^* = (n + i - 1) + 1 = n + i$ and $|\mathbf{p}^*| = |\mathbf{p}| + (n + i)$.

All that remains to show is that $|\lambda_1^* + \lambda_2^* + \cdots + \lambda_n^*| = |\lambda_1 + \lambda_2 + \cdots + \lambda_n| + (n + i)$. We will do this by writing each λ_k^* in terms of λ_k . Recall that for a given abacus, λ_k is the sum of the k -active positions which, in the notation of Lemma 4.1, can be written as $\lambda_k = \sum_{j=1}^k \#\{a \leq b_k : c(a) = j\}$.

By (c) above, $\lambda_k^* = \lambda_k$ for all k less than i . By (a) above, the largest part from the t -abacus is

$$\lambda_n^* = \lambda_i + 2n + \#\{a < b_i : c(a) > i\} = \lambda_i + 2n + \sum_{k=i+1}^n \#\{a < b_i : c(a) = k\}$$

where $c(a)$ refers to the class of position a in the $(t-1)$ -abacus. By (b) above, for each $k = i, \dots, n-1$ we have

$$\lambda_k^* = \lambda_{k+1} - \#\{a < b_k : c(a) = i\}$$

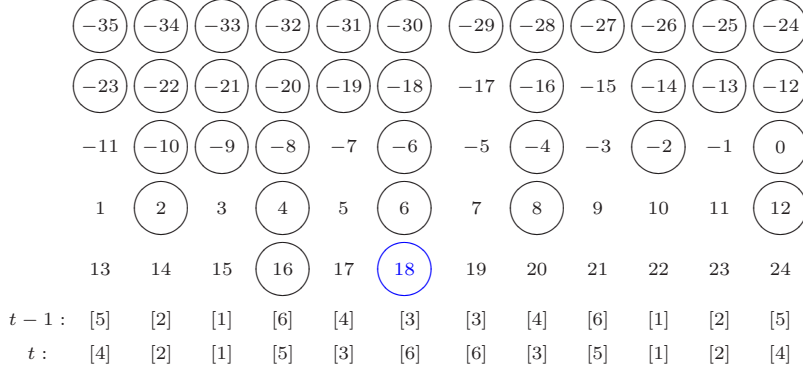
where once again $c(a)$ refers to the class of position a in the $(t-1)$ -abacus.

This implies that

$$|\lambda^*| - |\lambda| = 2n - \sum_{k=i+1}^n \left(\underbrace{\#\{a < b_k : c(a) = i\} - \#\{a < b_i : c(a) = k\}}_{=1 \text{ by Lemma 4.1}} \right)$$

and so $|\lambda^*| - |\lambda| = 2n - (n - i) = n + i$ as claimed. \square

Example 4.3. Consider the case of $t = 4$ for our running example in which we add a bead to position 18:



Then, the $(t-1)$ -abacus corresponds to

$$\lambda = (0, 1, 3, 6, 10, 16) \quad \text{and} \quad \mathbf{p} = (2, 4, 6, 7, 8, 9),$$

and the t -abacus corresponds to

$$\lambda^* = (0, 1, 6 - 2, 10 - 2, 16 - 2, 3 + 12 + 3) = (0, 1, 4, 8, 14, 18) \quad \text{and} \quad \mathbf{p}^* = (2, 4, 6, 7, 8, 9, 9).$$

Note that

$$|\lambda^*| - |\lambda| = (12 + 3) + (-2) + (-2) + (-2) = 9 = |\mathbf{p}^*| - |\mathbf{p}|.$$

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